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Traces of ghost orbits in the quantum standard map

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The semiclassical periodic orbit formula is compared with the asymptotic expansion of an exact analytical expression for the trace of the unitary propagator for the quantum standard map. Contributions arising from both real as well as ghost orbits, having complex classical action, have to be included to recover the usual semiclassical fixed point quantization in this limit as was observed by Kuś, Haake, and Delande [Phys. Rev. Lett. **71**, 2167 (1993)]. The range of validity of this expansion is derived and a way to extend its range is given.

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Recent work on what has been referred to as "post-modern quantum mechanics" [1] has focused on the influence of unstable periodic orbits on the spectral features of quantum dynamics. These orbits contribute to the eigenvalue spectrum of the Hamiltonian via Gutzwiller's trace formula [2] but they also leave their mark directly on the eigenstates giving rise to the phenomenon of scars [3–5]. Recently it was found [6], for the kicked top, that even imaginary or "ghost" orbits (which turn into real periodic orbits through a tangent bifurcation) are important for the spectrum of the unitary, single-kick propagator U. As the effect is seen in the trace of U the semiclassical analysis can be restricted to period-1 fixed points.

In this paper, we demonstrate the inherent simplicity of the semiclassical quantization for kicked dynamics and illustrate the generic features by considering the paradigmatic standard map. With variables q and p denoting the position and momentum of a particle, it is well known that the dynamics is equivalent to the following map:

$$\begin{aligned} q(t+1) &= q(t) + p(t+1) \;, \\ p(t+1) &= p(t) + K \sin q(t) \;, \end{aligned} \tag{1}$$

with integer time t. Taking q and p modulo 2π means the classical map has a toroidal phase space.

The corresponding dynamics of the quantum kicked rotor or standard map [7] is given by

$$\psi(t+1) = U\psi(t), \tag{2}$$

where the unitary evolution operator over a single kick is

$$U(t) = e^{-ip^2/2\hbar} e^{-iV(q)/\hbar} \;, \tag{3}$$

with the kicking potential $V(q) = K\cos(q)$ and the kicking constant $K > 0$.

For $\hbar = 2\pi/N$ with integer N the quantum system becomes periodic in p as well and the infinite-dimensional Hilbert space can be broken down into a d-dimensional Hilbert space (where $d = N$ for N even, $d = 2N$ for N odd) with appropriate boundary conditions. For even N every eigenvector ψ of U in p representation can be related to an N-component vector $\psi(b)$:

$$\psi_{n+Nl} = e^{-ibl}\psi(b)_n \;, \tag{4}$$

where the "Bloch number" b takes values between 0 and 2π, with $1 \le n \le N$ and $-\infty \le l \le \infty$ (for details see [8]). The eigenvalues of U can be found by fixing b and then diagonalizing the unitary matrix

$$U_{mn} = \frac{1}{N} e^{-i\hbar m^2/2} \sum_{j=1}^{N} e^{i(n-m)\theta_j} e^{-iK\cos(\theta_j)/\hbar} \;, \tag{5}$$

with $\theta_j = (2\pi j + b)/N$. (For odd N the exact form of the $2N \times 2N$ matrix U_{mn} is also known [8].) For our purposes here, we will consider only even N.

In this case, the trace of the $N \times N$ matrix U given by Eq. (5) can be rewritten in an instructive way. We note that

$$\mathrm{tr}U = \frac{1}{N} \sum_{n=1}^{N} e^{-i\pi n^2/N} \sum_{j=1}^{N} e^{-iz\cos(\theta_j)} \;, \tag{6}$$

with $z = KN/2\pi$. The first sum over n can be related to a Gauss sum (see, for example, [9]) and gives $\sqrt{N/i}$. The second sum can be expanded in terms of Bessel functions using the identity (see [10])

$$e^{-iz\cos(\theta_j)} = J_0(z) + 2\sum_{k=1}^{\infty} (-i)^k J_k(z)\cos(k\theta_j) \;. \tag{7}$$

The sum over j of $\cos(k\theta_j)$ vanishes except when k is a positive multiple of N, leading to a factor $N\sum_\mu \delta_{k,\mu N}\cos(\mu b)$ (where $\delta_{k,\mu N}$ denotes the Kronecker delta with positive integer μ) which reduces Eq. (6) to

$$\mathrm{tr}U = \sqrt{N/i}\left(J_0(z) + 2\sum_{\mu=1}^{\infty} J_{\mu N}(z) e^{-\frac{i\pi\mu N}{2}} \cos(\mu b) \right) \;. \tag{8}$$

Our result can also be motivated by recognizing that the toroidal geometry of the phase space leads to

$$\mathrm{tr}U = \sum_{m=-\infty}^{\infty} e^{imb} \int_{-\pi}^{\pi} dp\, \langle p + 2m\pi|U|p\rangle \ , \tag{9}$$

with momentum eigenstates $|p\rangle$. The sum is now over contributions that wrap around the torus m times each giving rise to the phase factor e^{imb}. Performing the integration leads to the result stated in Eq. (8). Note that for odd N the expansion has a prefactor with modulus of order $\sqrt{N}$ but is otherwise of the same form as that for even N. In the following we consider only Bloch number $b = 0$.

A number of interesting inferences can be drawn from this simple expression, which is the main focus of this paper. In Eq. (8) only a finite number of terms contribute as for $\mu N \leq z = KN/2\pi$ all terms with $\mu > K/2\pi$ are exponentially small. This is closely paralleled by the behavior of the classical map. Each time K becomes larger than $2|m|\pi$ with integer m, orbits occur which wrap around in the p direction m times. The classical map then develops a new pair of "accelerator modes" [11] which are period-1 fixed points with $p = 0$ and q determined by the solution of

$$\sin(q_m^*) = 2\pi m/K \ . \tag{10}$$

For $K > 2\pi|m|$ Eq. (10) has a pair of real solutions for m positive and negative at $q^*_{|m|} = \pi/2 \pm \arccos(2\pi|m|/K)$ and $q^*_{-|m|} = 3\pi/2 \pm \arccos(2\pi|m|/K)$. But for $K < 2\pi|m|$ these become "ghost orbits" in the complex plane at $q^*_{|m|} = \pi/2 \pm i\,\mathrm{arccosh}(2\pi|m|/K)$ and $q^*_{-|m|} = 3\pi/2 \pm i\,\mathrm{arccosh}(2\pi|m|/K)$.

We can now construct the semiclassical approximation to tr$U$ by summing the contributions from all fixed points $(q_\alpha, p = 0)$ [12] including accelerator modes and ghost orbits in

$$\begin{aligned}(\mathrm{tr}U)_{\mathrm{sc}} &= \sum_\alpha \frac{e^{iS(q_\alpha,q_\alpha)/\hbar}}{\sqrt{-V''(q_\alpha)}} \\ &= \sum_\alpha \frac{1}{\sqrt{|V''(q_\alpha)|}} e^{iS(q_\alpha,q_\alpha)/\hbar - i\nu_\alpha\pi/2 - i\gamma_\alpha\pi/4} \ ,\end{aligned} \tag{11}$$

where ν_α are the Maslov indices and γ_α are the ghost orbit indices which vanish for real orbits.

To evaluate the action near an accelerator mode or ghost orbit $(q_m^*, 0)$ we use the action on the cylindrical phase space (no periodicity in the p direction) and undo the $2\pi m$ shift in the p direction explicitly leading to

$$S_m(q,q') = \frac{1}{2}(q - q')^2 - K\cos(q) - 2\pi m q' \ . \tag{12}$$

The classical map follows from $p = -\partial S_m/\partial q$ and $p' = \partial S_m/\partial q'$.

There are two primary fixed points at $(q,p) = (0,0)$ and $(\pi, 0)$ which are hyperbolic ($j = 1$) and elliptic ($j = 2$), respectively. Their actions are $S_0^{(j)} = (-1)^j K$ with Maslov indices $\nu = 0$ and 1 for $j = 1, 2$.

For $m > 0$ and $K \geq K_m = 2\pi m$ we get two new period-1 fixed points with actions

$$S_m^{(j)} = -m\pi^2 \mp \sqrt{K^2 - K_m^2} \pm 2\pi m \arccos(K_m/K) \ , \tag{13}$$

and Maslov indices $\nu = 0$ and 1. Similarly for $m > 0$ but $K < K_m$ we have

$$S_m^{(j)} = -m\pi^2 \mp i\sqrt{K_m^2 - K^2} \pm 2\pi i m\, \mathrm{arccosh}(K_m/K). \tag{14}$$

The actions for ghost orbits are complex [6] and it turns out that only the solution with $j = 1$ and the upper sign is physical. The corresponding Maslov index $\nu = 0$ and the ghost orbit index $\gamma = 1$. In all cases the actions for $m < 0$ are the same (up to a constant) as for $m > 0$ giving identical contributions to the orbit sum (11). The contributions of the new period-1 orbits to Eq. (11) are completed by calculating the prefactors $1/\sqrt{|V''(q)|} = |K_m^2 - K^2|^{-1/4}$.

How does this periodic orbit expansion compare with Eq. (8) in the semiclassical limit $N \to \infty$? Using well known asymptotic formulas for Bessel functions of both large argument $z = KN/2\pi$ and order $mN > 0$ [10] we can approximate the Bessel functions in Eq. (8) by

$$\sqrt{N/i}\, J_0(z) \sim K^{-1/2}\left(e^{iS_0^{(1)}/\hbar} - ie^{iS_0^{(2)}/\hbar}\right) \ . \tag{15}$$

For $K > K_m$:

$$\sqrt{N/i}\, J_{mN}(z)\, e^{-i\pi m N/2} \sim \frac{e^{iS_m^{(1)}/\hbar} - ie^{iS_m^{(2)}/\hbar}}{(K^2 - K_m^2)^{1/4}} \ , \tag{16}$$

while for $K < K_m$:

$$\sqrt{N/i}\, J_{mN}(z)\, e^{-i\pi m N/2} \sim \frac{e^{iS_m^{(1)}/\hbar - i\pi/4}}{(K_m^2 - K^2)^{1/4}} \ , \tag{17}$$

with the classical actions specified in Eqs. (13) and (14) and in the text above. This reproduces the semiclassical trace formula (11) exactly including the Maslov indices. But it also illustrates that the fixed point expansion has to be carried out over both real and complex fixed points. Furthermore, it shows that only the ghost orbit with Maslov index $\nu = 0$ contributes to the semiclassical sum (11).

For the rest of this paper we will analyze the expression in Eq. (8) more carefully. We Fourier transform the exact trace of $U(N)$ with respect to $n = N/2 = \pi/\hbar$ appropriate for even N:

$$A(\omega) = \frac{1}{\Delta n} \sum_{n=n_0}^{n_0+\Delta n-1} e^{-in\omega}\, \mathrm{tr}U(2n) \ . \tag{18}$$

We can now compare the peak position ω with the real part of the classical actions $S_m^{(j)}$ and the peak amplitude $A(\omega)$ with the semiclassical amplitude in Eq. (11) which depends on both $|V''(q_\alpha)|$ and the (positive) imaginary part of $S_m^{(j)}$ (see also [6]). Figure 1 contrasts spectra with and without ghost peaks using the exact trace and the expansion Eq. (8) where only J_0 and J_N have been in-

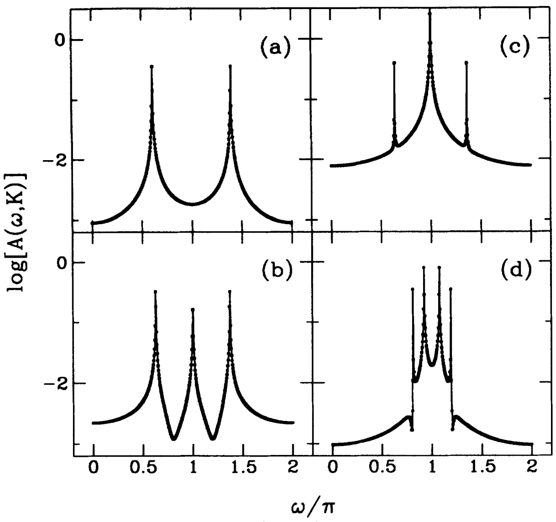

FIG. 1. Fourier spectra $A(\omega, K)$ as defined in Eq. (18) for $n_0 = 500$ and $\Delta n = 501$, with (a) $K = 6.0$, (b) $K = 6.2$, (c) $K = 6.3$, and (d) $K = 8.0$. The full line is the spectrum using the exact trace while points are obtained using the first two terms in expansion Eq. (8). The two sets of data agree within the resolution of the plot. The ghost peak is seen at $\omega/\pi = 1$ in panel (b). (The base of the logarithm in the ordinate label is 10.)

cluded. As anticipated, for $K \approx 2\pi$, the two primary fixed points as well as the ghost orbits or accelerator modes give the main contribution to the Fourier spectrum.

In Fig. 2 we show the K dependence of the amplitude of the "ghost peak" that appears at $\omega = \pi$ for K slightly below 2π, becomes associated with an accelerator mode at $K_1 = 2\pi$ [see Fig. 1(b)], and finally splits into two peaks for $K > 2\pi$ [compare Figs. 1(c) and 1(d)]. The peak amplitude of the Fourier spectrum using exact traces with even N shows perfect agreement with the corresponding amplitude using only the $\mu = 1$ term in the trace expansion Eq. (8).

The asymptotic (large N) behavior of the height of the new peak can be obtained by using the large order and

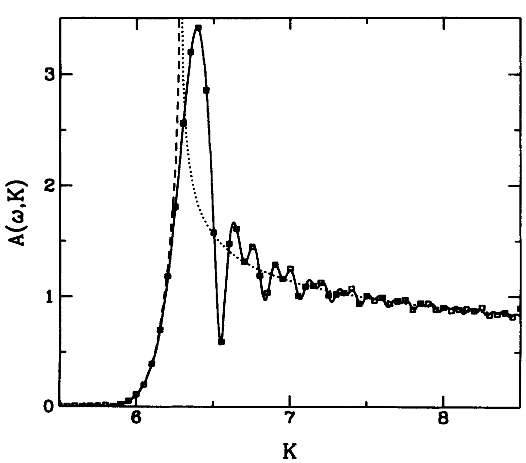

FIG. 2. The amplitude of the Fourier peak near $\omega = \pi$ as a function of K. Shown are the exact values (points), the Fourier transform using the $\mu = 1$ term in Eq. (8), and the two asymptotic curves (19) and (20).

large argument approximations to the Bessel functions in evaluating Eq. (18) at the peak position. Thus we find for $K < 2\pi$ the ghost peak amplitude to be

$$A_{\text{ghost}} = 2(4\pi^2 - K^2)^{-1/4} \, \frac{e^{-2\beta n_0}\left(1 - e^{-2\beta\, \Delta n}\right)}{\Delta n \, \left(1 - e^{-2\beta}\right)} \, , \quad (19)$$

where $\beta = \operatorname{arccosh}(2\pi/K) - \sqrt{1 - (K/2\pi)^2} > 0$ is proportional to the imaginary part of the action $S_1^{(1)}$ of the ghost orbit. For the real orbit ($K > 2\pi$)

$$A_{\text{acc}} = 2(K^2 - 4\pi^2)^{-1/4} \quad (20)$$

is the corresponding amplitude. Both these curves are shown in Fig. 2. A closer inspection shows the growth of the peak starting well below the critical K_1 but reaching the maximum amplitude only for K well above K_1.

For large values of N the ghost peak with index m is visible only for K smaller than but sufficiently close to K_m. How close depends on the imaginary part of the orbit's action which when multiplied by N gives the damping term for the amplitude. We find to leading order:

$$\text{Im}\, S_m^{(1)} \propto (K_m - K)^{3/2} \quad (21)$$

in agreement with the result for the kicked top [6]. This explains the prominence of ghost peaks even in the semiclassical limit of rather large values of N and for K values relatively far away from the bifurcation.

In the spirit of the paper by Kuś, Haake, and Delande [6] the interpolation between the stationary phase approximations for $J_{mN}(KN/2\pi)$ given in Eqs. (16) and (17) for K on either side of K_m can be improved in terms of the Airy function Ai (see p. 366 of [10]). From the argument of the Airy function one concludes that the range of K around K_m in which this improved approximation has to be used rather than the standard Gutzwiller approximation is given by

$$|K - K_m| < \Delta K \approx (mN)^{-2/3} \, , \quad (22)$$

which parallels the result for the kicked top [6]. This agreement indicates that the occurrence of spectral structures related to ghost orbits is the rule rather than the exception. Recently a ghost peak has been observed in experiments with helium Rydberg states in a magnetic field [13]. Also it is probable that some of the older results on hydrogen in a magnetic field can be explained by considering ghost orbits.

Our results here provoke a number of questions. Numerical evidence shows that ghost orbits can also provide support for scars, just like unstable periodic orbits. The details of this ghost orbit scarring and its experimental relevance are as yet unclear. An extension of our analysis to $\text{tr}(U^n)$ should also give interesting insights. For example, preliminary results show evidence of prebifurcation ghosts appearing in $\text{tr}(U^2)$ near $K = 4$ for the standard map. However, a systematic investigation would be restricted to maps having a coding for their periodic orbits like the piecewise linear standard map [4,5]. Furthermore, the quantum kicked rotor is known to show dynamical localization for irrational values of $\hbar/\pi$, leading to localized eigenfunctions and a discrete spectrum

[7]. Rational approximations of the form $\hbar = 2\pi M/N$ can be used to approach this limit. For $M < N \gg 1$ but $M^2 \gg N$ one expects the eigenphases to be insensitive to changes of the boundary conditions, i.e., the Bloch number b. This avenue of reasoning would address the issue of whether the phenomenon of dynamical localization can be explained within the context of periodic orbit quantization.

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